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# Nonregular Graphs with Three Eigenvalues

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We study nonregular graphs with three eigenvalues. We determine all the ones with least eigenvalue −2, and give new infinite families of examples. © 1998



## 1. INTRODUCTION

In this paper we look at the graphs that are generalizations of strongly regular graphs (cf. [3, 6, 16]) by dropping the regularity condition. More specifically, we consider graphs of which the adjacency matrices have three distinct eigenvalues. Seidel (cf. [15]) did a similar thing for the Seidel matrix by introducing strong graphs, which turned out to have an easy combinatorial characterization. Similarly, a nice combinatorial characterization is found for graphs with three Laplace eigenvalues [9]. The problem of graphs with *few* eigenvalues was perhaps first raised by Doob [11]. For more on such graphs we refer to [8].

Regular graphs with three eigenvalues are well-known to be strongly regular. Therefore we focus on the nonregular graphs, with three (adjacency) eigenvalues. Here the combinatorial simplicity seems to disappear with the regularity. This all lies in the algebraic consequence that the all-one vector is no longer an eigenvector. The Perron–Frobenius eigenvector still contains a lot of information, but the problem is to derive this eigenvector from the spectrum.

Earlier results on nonregular graphs with three eigenvalues were found by Bridges and Mena [2] and Muzychuk and Klin [14]. In this paper we determine all the ones with least eigenvalue $-2$, and give new families of examples by using symmetric designs, affine designs, antipodal covers of the complete graph, and systems of linked symmetric designs.



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## 2. EXAMPLES

A large family of (in general) nonregular examples is given by the complete bipartite graphs $K_{m,n}$ with spectrum $\{[\sqrt{mn}]^1, [0]^{m+n-2}, [-\sqrt{mn}]^1\}$. Note that a connected graph with three eigenvalues has diameter two, hence a bipartite one must be complete bipartite. Other examples were found by Bridges and Mena [2] and Muzychuk and Klin [14], most of them being cones.

### 2.1. *Cones*

A *cone* over a graph $G$ is obtained by adding a vertex to $G$ that is adjacent to all vertices of $G$. If $G$ is a strongly regular graph on $v$ vertices, with degree $k$ and smallest eigenvalue $s$, then the cone over $G$ is a graph with three eigenvalues if and only if $s(k-s)=-v$ (cf. [14]). This condition is satisfied by infinitely many strongly regular graphs, which implies that there are infinitely many cones with three eigenvalues. A small example is given by the cone over the Petersen graph, with spectrum $\{[5]^1, [1]^5, [-2]^5\}$ (see Fig. 1). Bridges and Mena [2] obtained results on cones with distinct eigenvalues $\theta_0$, $\theta_1$ and $-\theta_1$. They proved that such graphs are cones over strongly regular $(v, k, \lambda, \lambda)$ graphs with three possible exceptions (for the spectra): There is an example with spectrum $\{[21]^1, [3]^{19}, [-3]^{26}\}$, an example with spectrum $\{[56]^1, [4]^{41}, [-4]^{55}\}$, and the case with spectrum $\{[204]^1, [6]^{127}, [-6]^{161}\}$ is open.

### 2.2. *Switching in Strongly Regular Graphs*

Other examples can be constructed by switching in a strongly regular graph. Switching with respect to some subset of the vertices means that we interchange the edges and nonedges between the subset and its complement. Muzychuk and Klin [14] found parameter conditions for switching in a strongly regular graph to obtain a nonregular graph with three eigenvalues.

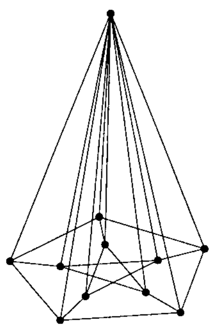

**FIG. 1.** The cone over the Petersen graph.

Moreover, they proved that the only graph that can be obtained from switching in a triangular graph is the one obtained by switching in $T(9)$ with respect to an 8-clique. This gives a graph with spectrum $\{[21]^1, [5]^7, [-2]^{28}\}$.

We find new examples by switching with respect to an 8-clique in the strongly regular graph that is obtained from a polarity in Higman's symmetric 2-(176, 50, 14) design, and by switching with respect to three vertex-disjoint 6-cliques in the strongly regular Zara graph with parameters (126, 45, 12, 18), and by switching with respect to 8 vertex-disjoint 8-cliques in the point graph (with parameters (512, 196, 60, 84)) of the partial geometry $pg(7, 27, 3)$. Also, switching with respect to two vertex-disjoint 16-cliques in the Latin square graph $L_7(16)$, 5 vertex-disjoint 25-cliques in $L_{12}(25)$, 6 vertex-disjoint 27-cocliques in $L_{15}(27)$, 27 vertex-disjoint 243-cliques in $L_{120}(243)$ and 76 vertex-disjoint 361-cliques in $L_{180}(361)$ give examples of nonregular graphs with three eigenvalues. A more complicated example is constructed by Martin [private communication]. Take the strongly regular (105, 72, 51, 45) graph on the flags (incident point-line pairs) of $PG(2, 4)$, where two distinct flags $(p_1, l_1)$ and $(p_2, l_2)$ are adjacent if $p_1 = p_2$ or $l_1 = l_2$ or ($p_1 \notin l_2$ and $p_2 \notin l_1$). Now switching with respect to a set of 21 flags with the property that every point and every line is in precisely one flag (such a set exists by elementary combinatorial theory since it corresponds to a perfect matching in the incidence graph of $PG(2, 4)$) yields a nonregular graph with spectrum $\{[60]^1, [9]^{21}, [-3]^{83}\}$.

## 2.3. *Symmetric and Affine Designs*

A new family of nonregular graphs with three eigenvalues is constructed from symmetric 2-$(q^3 - q + 1, q^2, q)$ designs. Such designs exist if $q$ is a prime power and $q - 1$ is the order of a projective plane (cf. [1]). Take the bipartite incidence graph of such a design and add edges between all blocks. The resulting graph has spectrum $\{[q^3]^1, [q-1]^{(q-1)q(q+1)}, [-q]^{(q-1)q(q+1)+1}\}$. The smallest example is derived from the complement of the Fano plane, and we find a graph with spectrum $\{[8]^1, [1]^6, [-2]^7\}$. (See Fig. 2.) The

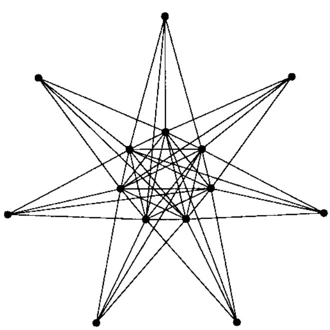

**FIG. 2.** The graph derived from the complement of the Fano plane.

next case comes from 2-(25, 9, 3) designs. Denniston [10] found that there are exactly 78 such designs, and so there are at least 78 graphs with spectrum $\{[27]^1, [2]^{24}, [-3]^{25}\}$. By using the same methods as in the proof of Theorem 7, we were able to show that these are all the graphs with this spectrum.

Next, consider the design on the points and planes of the affine geometry $AG(3, q)$ of dimension 3 over the field with $q$ elements. Take the bipartite incidence graph of this design, and add an edge between two planes if they intersect (in $q$ points). This graph is nonregular with three eigenvalues. In fact, for any affine 2-$(q^3, q^2, q+1)$ design this construction gives a graph with spectrum $\{[q^3+q^2+q]^1, [q]^{q^3-1}, [-q]^{q^3+q^2+q}\}$. The smallest example ($q=2$) of this infinite family is on 22 vertices with spectrum $\{[14]^1, [2]^7, [-2]^{14}\}$, and was already constructed by Bridges and Mena [2]. Note that the graph derived from the complement of the Fano plane is an induced subgraph of this graph.

## 3. NONINTEGRAL EIGENVALUES

For all known nonregular examples, except for the complete bipartite ones, all eigenvalues are integral. We shall prove that the only graphs with three eigenvalues for which the largest eigenvalue is not integral, are the complete bipartite graphs. Without proof we give the following useful lemma (cf. [7, Theorem 3.6, 3.8]).

LEMMA 1. *Let $G$ be a connected graph on $v$ vertices, with eigenvalues $\theta_i$ and corresponding multiplicities $m_i$, and with largest eigenvalue $\theta_0$. Denote by $k_{\max}$ and $k_{\text{ave}}$ the largest and average vertex degree, respectively. Then $vk_{\text{ave}} = \sum_i m_i \theta_i^2 \leqslant v\theta_0$ with equality if and only if $G$ is regular. Moreover, $\theta_0 \leqslant k_{\max}$, also with equality if and only if $G$ is regular.*

PROPOSITION 2. *Let $G$ be a connected graph with three distinct eigenvalues of which the largest is not an integer. Then $G$ is a complete bipartite graph.*

*Proof.* Suppose $G$ has $v$ vertices, adjacency matrix $A$ with largest eigenvalue $\theta_0$ and remaining eigenvalues $\theta_1$ and $\theta_2$. Since $\theta_0$ is simple and not integral, it follows that at least one of these remaining eigenvalues is also simple and nonintegral. If we have only three vertices, then there is only one connected, noncomplete graph: $K_{1,2}$. So we may assume $G$ to have more than three vertices. In this case the remaining eigenvalue is of course not simple, and it follows that $\theta_0$ and say $\theta_2$ are of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b$ integral, and then $\theta_1$ is integral. Moreover, since $\theta_2 \geqslant -\theta_0$, we must have $a \geqslant 0$. Since the adjacency matrix of $G$ has zero trace, it follows that $a+(v-2)\,\theta_1=0$, so $a$ is a multiple of $v-2$, and $\theta_1 \leqslant 0$.

If $a=\theta_1=0$, then $G$ is bipartite, hence complete bipartite. If $\theta_1=-1$, then $a=v-2$, and it follows that $-1$ is the smallest eigenvalue of $G$, otherwise we would have $\theta_0>v-1$, which is a contradiction. But then $A+I$ is a positive semidefinite matrix of rank two, and we would have that $G$ is the disjoint union of two cliques, which is again a contradiction. If $\theta_1=-2$, then $a=2(v-2)$, and it follows that $-2$ is the smallest eigenvalue of $G$. Now $A+2I$ is positive semidefinite of rank two, which is easily seen to be impossible (for example by examining the possible principal minors of order 3). For the remaining case we have $\theta \leqslant -3$, and then $a \geqslant 3(v-2)$. From $\theta_0 \leqslant v-1$, we now find that we can have at most three vertices, and so the proof is finished. ∎

So if we have a graph with three eigenvalues, which is not a complete bipartite graph, then we know that its largest eigenvalue is integral. The remaining two eigenvalues, however, can still be nonintegral, with many of the strongly regular conference graphs as examples. The following proposition reflects what is known as the "half-case" for strongly regular graphs.

PROPOSITION 3. *Let $G$ be a connected graph on $v$ vertices with three eigenvalues $\theta_0>\theta_1>\theta_2$, which is not a complete bipartite graph. If not all eigenvalues are integral, then $v$ is odd and $\theta_0=\frac{1}{2}(v-1)$, $\theta_1, \theta_2=-\frac{1}{2}\pm\frac{1}{2}\sqrt{b}$ for some $b\equiv 1 \pmod 4$, $b\leqslant v$, with equality if and only if $G$ is strongly regular. Moreover, if $v\equiv 1 \pmod 4$ then all vertex degrees are even, and if $v\equiv 3 \pmod 4$ then $b\equiv 1 \pmod 8$.*

*Proof.* According to the previous proposition $\theta_0$ is integral, so $\theta_1$ and $\theta_2$ must be of the form $\frac{1}{2}(a\pm\sqrt{b})$, with $a, b$ integral, with the same multiplicity $\frac{1}{2}(v-1)$. Since the adjacency matrix has zero trace, we have $\theta_0+\frac{1}{2}a(v-1)=0$. Since $0<\theta_0<v-1$, and $a$ is integral, we must have $\theta_0=\frac{1}{2}(v-1)$, $a=-1$. Now $\theta_1\theta_2$ is integral, and it follows that $b\equiv 1 \pmod 4$. Moreover, by Lemma 1 we have that the average vertex degree $k_{\text{ave}}=(\theta_0^2+\frac{1}{2}(v-1)(\theta_1^2+\theta_2^2))/v=\frac{1}{2}\theta_0(v+b)/v$ is at most $\theta_0$, with equality if and only if $G$ is strongly regular. This inequality reduces to $b\leqslant v$.

From the equation $(A-\theta_0 I)(A-\theta_1 I)(A-\theta_2 I)=O$, we find that

$$A^3=\tfrac{1}{2}(v-3)\,A^2+(\tfrac{1}{2}(v-1)+\tfrac{1}{4}(b-1))\,A-\tfrac{1}{8}(v-1)(b-1)I.$$

The diagonal element of this matrix corresponding to vertex $x$ counts twice the number of triangles $\Delta_x$ through $x$. Thus we find that

$$\Delta_x=\tfrac{1}{4}(v-3)\,d_x-\tfrac{1}{16}(v-1)(b-1),$$

where $d_x$ is the vertex degree of $x$. Since $\Delta_x$ is integral we find that if $v \equiv 1 \pmod 4$, then $d_x$ must be even, for every vertex $x$. If $v \equiv 3 \pmod 4$, then we must have $b \equiv 1 \pmod 8$. ∎

Of course, also if the eigenvalues are integral, we find restrictions for the degrees from the expression for $\Delta_x$.

PROPOSITION 4. *Let $G$ be a graph with three integral eigenvalues $\theta_0$, $\theta_1$ and $\theta_2$. If all three eigenvalues are odd, then all vertex degrees are odd. If one of them is odd and two are even, then all vertex degrees are even. Moreover, if a vertex $x$ has degree $d_x$, then the number of triangles through $x$ is given by $\Delta_x = \frac{1}{2}(\theta_0 + \theta_1 + \theta_2)\, d_x + \frac{1}{2}\theta_0\theta_1\theta_2$.*

Returning to the graphs of Proposition 3, we should mention that we do not know any nonregular example. Although the cone over the Petersen graph (with $v = 11, b = 9$) is nonregular, its eigenvalues are integral. Also, the nonbipartite graphs with nonintegral eigenvalues must have at least three distinct vertex degrees, as we shall see in the next section.

## 4. THE PERRON–FROBENIUS EIGENVECTOR

An important property of connected graphs with three eigenvalues is that $(A - \theta_1 I)(A - \theta_2 I)$ is a rank one matrix. It follows that we can write

$$(A - \theta_1 I)(A - \theta_2 I) = \alpha\alpha^T, \qquad \text{with} \quad A\alpha = \theta_0 \alpha.$$

Moreover, from the Perron–Frobenius theorem it follows that the Perron–Frobenius eigenvector $\alpha$ is a positive eigenvector, that is, all its components are positive. From the quadratic equation we derive that

| | |
|---|---|
| $d_x = -\theta_1\theta_2 + \alpha_x^{\,2}$ | is the degree of vertex $x$, |
| $\lambda_{x,y} = \theta_1 + \theta_2 + \alpha_x\alpha_y$ | is the number of common neighbors of $x$ and $y$, if they are adjacent, |
| $\mu_{x,y} = \alpha_x\alpha_y$ | is the number of common neighbors of $x$ and $y$, if they are not adjacent. |

If we assume $G$ not to be complete bipartite, so that $\theta_1 + \theta_2$ and $\theta_1\theta_2$ are integral, it follows that $\alpha_x\alpha_y$ is an integer for all $x$ and $y$. We immediately see that this imposes strong restrictions for the possible degrees that can occur. We also see that if the graph is regular, then we have a strongly regular graph.

### 4.1. *Two Vertex Degrees*

Now suppose that $G$ has only two vertex degrees (which is the case in most known nonregular examples), say $k_1$ and $k_2$, with respective $\alpha_1$ and $\alpha_2$. Now fix a vertex $x$ of degree $k_1$. Let $k_{1,1}$ and $k_{1,2}$ be the numbers of vertices of degree $k_1$ and $k_2$, respectively, that are adjacent to $x$. Then it follows that $k_{1,1}+k_{1,2}=k_1$ and since $A\alpha=\theta_0\alpha$, it follows that $k_{1,1}\alpha_1+k_{1,2}\alpha_2=\theta_0\alpha_1$. These two equations uniquely determine $k_{1,1}$ and $k_{1,2}$, and the solution is independent of the chosen vertex $x$ of degree $k_1$. Similarly we find $k_{2,1}$ and $k_{2,2}$, and it follows that the partition of the vertices according to their degrees is regular. This imposes the restriction $n_1k_{1,2}=n_2k_{2,1}$, where $n_i$ is the number of vertices of degree $k_i$. Note that the equations $n_1+n_2=v$, and $n_1k_1+n_2k_2=\sum_i m_i\theta_i^2$, determine $n_1$ and $n_2$ from the spectrum and $k_1$ and $k_2$. Note also that the quotient matrix with respect to the regular partition has an eigenvalue $\theta_0$, and hence also another integral eigenvalue $(k_{1,1}+k_{2,2}-\theta_0)$. This implies that also the graph has this other integral eigenvalue (with eigenvector $(k_{1,2}\alpha_2, ..., k_{1,2}\alpha_2, -k_{2,1}\alpha_1, ..., -k_{2,1}\alpha_1)^T$, the first part corresponding to the vertices of degree $k_1$). Thus we find that if $G$ is nonbipartite with non-integral eigenvalues, then it has at least three distinct vertex degrees.

### 4.2. *Cones*

If we assume that $G$ is a cone, say over $H$, then we can prove that we have at most three distinct degrees. Indeed, take a vertex with degree $k_1=v-1$, with corresponding $\alpha_1$, and suppose we have another vertex of degree $k_i$ (with corresponding $\alpha_i$). Now the common neighbours of these vertices are $k_i-1$ in number, so $k_i-1=\lambda_{1,i}=\alpha_1\alpha_i+\theta_1+\theta_2$. But also $k_i=-\theta_1\theta_2+\alpha_i^2$, and so we get a quadratic equation for $k_i$, and so $k_i$ can take at most two values. If $k_i$ takes only one value, then we easily see that $H$ must be strongly regular. If we have precisely two other degrees, say $k_2, k_3$, with respective $\alpha_2, \alpha_3$, then it follows from the quadratic equation that $\alpha_1=\alpha_2+\alpha_3$. Here it also follows quite easily that the partition of the vertices according to their degrees is regular. Bridges and Mena [2] used this to show that there are only three parameter sets for cones with eigenvalues $\theta_0$, $\pm\theta_1$ over a nonregular graph.

### 4.3. *Three Vertex Degrees*

When we are trying to construct, or prove nonexistence of a graph $G$ with given spectrum, and with possible vertex degrees $k_i$ (and corresponding $\alpha_i$), we have the following restrictions. If $n_i$ is the number of vertices of degree $k_i$, then first of all we have the restrictions $\sum_i n_i=v$, and $\sum_i n_ik_i=\sum_i m_i\theta_i^2$, as we have seen before. Another restriction is obtained by adding the entries in the vector equation $A\alpha=\theta_0\alpha$. Thus we obtain $\sum_i n_ik_i\alpha_i=\underline{1}^TA\alpha=\underline{1}^T\theta_0\alpha=\theta_0\sum_i n_i\alpha_i$, hence $\sum_i n_i(k_i-\theta_0)\,\alpha_i=0$.

If $G$ has three possible vertex degrees $k_1$, $k_2$ and $k_3$, then the respective numbers of vertices $n_i$ are determined by the three equations given. The integrality of the numbers $n_i$ is a further obvious, but strong, restriction. Furthermore, we shall prove now that also here the partition of the vertices according to their degrees is regular. Let $k_{i,j}$ be the number of vertices of degree $j$ adjacent to a vertex of degree $i$. Fix $i$, then obviously $\sum_j k_{i,j} = k_i$, and by inspecting (an entry corresponding to a vertex of degree $k_i$ in) the equation $A\alpha = \theta_0 \alpha$ we find that $\sum_j \alpha_j k_{i,j} = \alpha_i \theta_0$. The third equation, which together with these two will determine all $k_{i,j}$ (in case of three vertex degrees), is derived from inspecting the equation $A^2\underline{1} = (\theta_1 + \theta_2) A\underline{1} - \theta_1\theta_2 I\underline{1} + \alpha\alpha^T\underline{1}$. Now we find that

$$\sum_j k_j k_{i,j} = (\theta_1 + \theta_2) k_i - \theta_1\theta_2 + \left(\sum_j n_j \alpha_j\right) \alpha_i.$$

Hence if we have only three degrees then the partition is regular. The $k_{i,j}$ thus found must of course be integral, and furthermore $n_i k_{i,j} = n_j k_{j,i}$ and $0 \leqslant k_{i,j} \leqslant n_j - \delta_{ij}$.

### 4.4. *Four Vertex Degrees*

Another restriction (the fourth) on the numbers $n_i$ is found by adding all entries in the matrix equation $(A - \theta_1 I)(A - \theta_2 I) = \alpha\alpha^T$. Thus we find that

$$\left(\sum_i n_i \alpha_i\right)^2 = \underline{1}^T \alpha\alpha^T \underline{1} = \underline{1}^T (A - \theta_1 I)(A - \theta_2 I)\, \underline{1} = \sum_i n_i (k_i - \theta_1)(k_i - \theta_2).$$

If $G$ has (at most) four vertex degrees, then after substitution of the other three equations for $n_i$ (from Section 4.3), this gives a quadratic equation for, say, $n_4$. Hence there are at most two possibilities for the numbers $n_i$. These observations will reduce the number of cases to be checked in the classification of graphs in the root system $E_8$ in the next section.

## 5. GRAPHS WITH LEAST EIGENVALUE $-2$

The results from Section 3 imply that the only connected graphs with three eigenvalues, all greater than $-2$ are the complete bipartite graphs $K_{1,2}$ and $K_{1,3}$ and graphs from Proposition 3 with $b = 5$. However, here we can only have the strongly regular 5-cycle $C_5$, which is the unique graph with spectrum $\{[2]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{5}]^2, [-\frac{1}{2} - \frac{1}{2}\sqrt{5}]^2\}$.

PROPOSITION 5. *If $G$ is a connected graph with three distinct eigenvalues, all greater than $-2$, then $G$ is either $K_{1,2}$, $K_{1,3}$, or $C_5$.*

*Proof.* By the previous remarks, besides $K_{1,2}$ and $K_{1,3}$ we only have to check spectra from Proposition 3 with $b=5$. First suppose $v>9$ (note that $v\equiv 1 \pmod 4$). Then for the average vertex degree we have $k_{\text{ave}}<\frac{1}{2}(v-1)$. Since the vertex degrees must be even, there must be a vertex $x$ of degree $d_x\leqslant\frac{1}{2}(v-1)-2$. If $d_x>2$, then for the number of triangles $\Delta_x$ through $x$ we have

$$\Delta_x=\frac{1}{4}(v-3)(d_x-1)-\frac{1}{2}>\frac{1}{4}(v-5)(d_x-1)\geqslant\binom{d_x}{2},$$

which is a contradiction. Also if $d_x=2$, then $\Delta_x=\frac{1}{4}(v-5)>2$, a contradiction. The case $v=9$ can be excluded by the following arguments, using the Perron–Frobenius eigenvector $\alpha$. Here it follows that there must be a vertex $x$ of degree 2, and so with $\alpha_x=1$. Now $\alpha$ is an integral vector, implying that the vertex degrees can only take values 2, 5, 10, .... But the vertex degrees must be even, and at most 8, so it follows that the graph is regular, which is a contradiction. ∎

Muzychuk and Klin [14] posed the problem of classifying all graphs with three eigenvalues, all of which are at least $-2$. By the characterization of Cameron, Goethals, Seidel, and Shult [5] (see also [6]), it follows that such a graph is a generalized line graph or can be represented by roots in the lattice $E_8$. Using this, we find the following.

THEOREM 6. *If $G$ is a connected graph with three distinct eigenvalues, all at least $-2$, then $G$ is isomorphic to one of $K_{1,2}, K_{1,3}, K_{1,4}$, $C_5$, $L_2(n)$, $n\geqslant 2$, $T(n)$, $n\geqslant 4$, or $CP(n)$, $n\geqslant 2$, or $G$ is represented by a subset of $E_8$.*

*Proof.* First, suppose that $G$ is a connected line graph, not $C_5$ or $K_{1,2}$, of some graph $H$, and $G$ has three eigenvalues, say $\theta_0>\theta_1>\theta_2=-2$. Here we may assume that $H$ is connected. Then the adjacency matrix $A$ of $G$ can be written as $A=N^TN-2I$, where $N$ is the vertex-edge incidence matrix of $H$. Now $NN^T=D+B$, where $D$ is the diagonal matrix of vertex degrees in $H$, and $B$ is the adjacency matrix of $H$. It follows that $D+B$ has eigenvalues $\theta_0+2$, $\theta_1+2$, and possibly 0. Suppose 0 is an eigenvalue with eigenvector $u$. Then $N^Tu=\underline{0}$. This implies that if $i$ and $j$ are adjacent in $H$, then $u_i=-u_j$. So $H$ is bipartite. Moreover, since $D+B$ has three distinct eigenvalues, it follows that $H$ has diameter at most two (the diameter is easily proven to be smaller than the number of distinct eigenvalues of $D+B$), so $H$ must be a complete bipartite graph $K_{m,n}$. Since the line graph of a nonregular complete bipartite graph has four distinct eigenvalues (unless $m$ or $n$ equals one, then we get a complete graph), $H$ must be the complete bipartite graph $K_{n,n}$, $n\geqslant 2$, with the lattice graph $L_2(n)$ as line graph. Now suppose that 0 is not an eigenvalue. Then $D+B$ has only two distinct eigenvalues, and it follows that $H$ is a complete graph $K_n$, with the triangular graph $T(n)$ as line graph.

Next, we assume that $G$ is a generalized line graph $L(H; a_1, ..., a_m)$ (where $m$ is the number of vertices of $H$), which is not a line graph, so at least one of the $a_i$ is nonzero. Now $G$ can be represented in $\mathbb{R}^n$, where $n = m + \sum_i a_i$, as follows. Take $\{e_{i,j} \mid i = 1, ..., m,\ j = 0, ..., a_i\}$ as an orthonormal basis of $\mathbb{R}^n$. Represent the vertices of $G$ by the vectors $e_{i,0} + e_{j,0}$ for all edges $\{i, j\}$ in $H$, and the vectors $e_{i,0} + e_{i,j}$ and $e_{i,0} - e_{i,j}$ for all $i = 1, ..., m$, $j = 1, ..., a_i$, any two of them being adjacent if and only if they have inner product one. In matrix form, if $N$ is the generalized $(0, \pm 1)$-incidence matrix, that is, with rows representing the basis of $\mathbb{R}^n$, and columns representing the vertices of $G$, then $A = N^T N - 2I$ is the adjacency matrix of $G$. Now suppose that $G$ has distinct eigenvalues $\theta_0 > \theta_1 > \theta_2 = -2$, then $NN^T$ has eigenvalues $\theta_0 + 2$, $\theta_1 + 2$, and possibly 0. Suppose that $NN^T u = \underline{0}$. Then $N^T u = \underline{0}$, so if $i$ and $j$ are adjacent in $H$, then $u_{i,0} = -u_{j,0}$. For $i$ with $a_i$ nonzero, we have that $u_{i,0} = -u_{i,j}$, and $u_{i,0} = u_{i,j}$ for $j = 1, ..., a_i$. So $u_{i,j} = 0$, and since we may assume $H$ to be connected, it follows that $u = \underline{0}$, so 0 is not an eigenvalue of $NN^T$. Now let us have a closer look at this matrix. After rearrangement of the rows of $N$, it follows that

$$NN^T = \begin{pmatrix} D + B & O \\ O & 2I \end{pmatrix},$$

where $D$ is the diagonal matrix with entries $D_{ii} = d_i + 2a_i$, where $d_i$ is the vertex degree of $i$ in $H$, and $B$ is the adjacency matrix of $H$. Thus it follows that $\theta_1 = 0$ and that $D + B$ has distinct eigenvalues $\theta_0 + 2$ (with multiplicity one) and (possibly) 2. If $H$ is a graph on one vertex, then there are no further restrictions, and $G$ then is a cocktail party graph $CP(n)$. Otherwise, $H$ is a complete graph (since the diameter is one), and since $D + B - 2I$ is a rank one matrix, it follows that $D = 3I$. Since $a_i$ is nonzero for some $i$, it now follows that $d_i = a_i = 1$ for all $i$. But then $H$ is a single edge, and $G$ is $K_{1,4}$. ∎

The strongly regular graphs with all eigenvalues at least $-2$ have already been classified by Seidel (cf. [5]). Besides the 5-cycle, the lattice graphs, the triangular graphs and the cocktail party graphs, there are the Petersen graph, the complement of the Clebsch graph, the Shrikhande graph, the complement of the Schläfli graph, and the three Chang graphs.

In Section 2 we already saw some nonregular graphs with three eigenvalues, all of which are at least $-2$. Besides these, we also mention the cones over the lattice graph $L_2(4)$, the Shrikhande graph, the triangular graph $T(8)$ and the three Chang graphs. Moreover, we have the following graph with spectrum $\{[11]^1, [3]^7, [-2]^{16}\}$, which was found by Spence [private communication], and which is strongly related to the strongly regular lattice graph $L_2(5)$ (which has spectrum $\{[8]^1, [3]^8, [-2]^{16}\}$). Fix a vertex $x$ in $L_2(5)$. Its set of neighbors can be partitioned into two 4-sets, each inducing a 4-clique. Now delete $x$, and (switch) interchange edges and nonedges between one of the 4-sets and the set of non-neighbors of $x$.

THEOREM 7. *The only connected graphs with three eigenvalues, all of which are at least* $-2$, *and which are not strongly regular or complete bipartite are the cone over the Petersen graph, the graph derived from the complement of the Fano plane* (*see Fig.* 2), *the cone over the Shrikhande graph, the cone over the lattice graph* $L_2(4)$, *the graph on the points and planes of* $AG(3, 2)$, *the graph related to the lattice graph* $L_2(5)$ (*see above*), *the cones over the Chang graphs, the cone over the triangular graph* $T(8)$, *and the graph obtained by switching in* $T(9)$ *with respect to an* 8-*clique*.

*Proof.* By Theorem 6 we only have to check the graphs that are represented in $E_8$. First we mention that a graph that is represented by a subset of $E_8$ has at most 36 vertices and vertex degrees at most 28, thus there are finitely many (see [4]). Note that both bounds are tight for the example obtained by switching in $T(9)$.

For graphs which are represented in the sublattice $E_6$, and which are not generalized line graphs, the multiplicity of the eigenvalue $-2$ is $v-6$, where $v$ is the number of vertices. Consequently such graphs have spectrum $\{[2(v-6)-5\theta_1]^1, [\theta_1]^5, [-2]^{v-6}\}$, and we also may assume that $\theta_1 \geqslant 1$ (a connected graph with exactly one positive eigenvalue must be a complete multipartite graph, cf. [7, Theorem 6.7]). Using the inequality $\sum_i m_i\theta_i^2 < v\theta_0$ (Lemma 1), we find that $\theta_1 = 1$ and $v = 11, 12, 13$.

For graphs which are represented in the sublattice $E_7$ (and which are not generalized line graphs or represented in $E_6$), the multiplicity of the eigenvalue $-2$ is $v-7$. Consequently they have spectrum $\{[2(v-7)-6\theta_1]^1, [\theta_1]^6, [-2]^{v-7}\}$. Here we find that $\theta_1 = 1$, 2 or 3. If $\theta_1 = 1$, then $v = 12, ..., 16$. In the case $\theta_1 = 2$, we find $v = 17, ..., 20$. If $\theta_1 = 3$, then $v = 22, 23, 24$.

For the remaining nonregular graphs represented by roots in $E_8$, we find possible spectra $\{[2(v-8)-7\theta_1]^1, [\theta_1]^7, [-2]^{v-8}\}$ with $\theta_1 = 1$ and $v = 13, ..., 19$, $\theta_1 = 2$ and $v = 18, ..., 25$, $\theta_1 = 3$ and $v = 23, ..., 30$, $\theta_1 = 4$ and $v = 29, ..., 35$, or $\theta_1 = 5$ and $v = 34, ..., 36$.

By computer we checked these spectra for possible vertex degrees, using the property that the product of any two entries of the Perron–Frobenius eigenvector $\alpha$ is integral. We checked that in a particular sequence of degrees, we have $k_{\min} < \theta_0 < k_{\max} \leqslant 28$, and we checked for a possible degree $d_x$ that the corresponding number of triangles $\Delta_x$ obtained from

Proposition 4 satisfies the condition $\Delta_x \leqslant \frac{1}{2}d_x(d_x-1)$. Of course, we also checked that all vertex degrees were even in the cases with two even eigenvalues (by Proposition 4). If in a particular sequence only two vertex degrees remained, then these were checked for the conditions which follow from the fact that in that case the vertices of distinct degrees induce a regular partition. Similarly we computed the numbers $n_i$ of degree $k_i$ from the equations in Sections 4.3 and 4.4, and checked their integrality, in case a sequence of degrees contained three or four degrees. If only three degrees remained, then these were also checked for the further conditions of Section 4.3. In none of the possible sequences of degrees more than four degrees appeared. What remains are the following possibilities. By a $*$ we denote that a graph exists. (See Table I.)

In $E_6$ there is one case, with $v=11$. Here we must have one vertex of degree 10 and ten vertices of degree 4. It is obvious that this must be the cone over the Petersen graph. In the first case in $E_7$, with $v=14$, we find seven vertices of degree 4 and seven vertices of degree 10. Moreover, the vertices of degree 4 induce a coclique, and the vertices of degree 10 induce a clique. Since any two vertices of degree 4 have two common neighbors (this follows from the Perron–Frobenius eigenvector), the edges between the vertices of degree 4 and the vertices of degree 10 form the incidence of the complement of the Fano plane, the unique 2-(7, 4, 2) design. So here we find the graph of Fig. 2, which is now proven to be the unique graph with spectrum $\{[8]^1, [1]^6, [-2]^7\}$. The other case in $E_7$, with $v=17$, must be a cone over a strongly regular graph with eigenvalues 6, 2 and $-2$, hence we must have the cone over the lattice graph $L_2(4)$ or the cone over the Shrikhande graph.

Now consider the case in $E_8$ with $v=22$. Here we have a regular partition, with 8 vertices of degree 7 and 14 vertices of degree 16. Furthermore, the

TABLE I

| $v$ | Spectrum | Degrees | |
|---|---|---|---|
| * 11 | $\{[5]^1, [1]^5, [-2]^5\}$ | 4 10 | $(n_4=10, n_{10}=1, k_{4,10}=1)$ |
| * 14 | $\{[8]^1, [1]^6, [-2]^7\}$ | 4 10 | $(n_4=7, n_{10}=7, k_{4,10}=4)$ |
| * 17 | $\{[8]^1, [2]^6, [-2]^{10}\}$ | 7 16 | $(n_7=16, n_{16}=1, k_{7,16}=1)$ |
| * 22 | $\{[14]^1, [2]^7, [-2]^{14}\}$ | 7 16 | $(n_7=8, n_{16}=14, k_{7,16}=7)$ |
| * 24 | $\{[11]^1, [3]^7, [-2]^{16}\}$ | 7 10 15 22 | $(n_7=4, n_{10}=16, n_{15}=4, n_{22}=0,$ $k_{7,10}=4,\ k_{7,15}=0,\ k_{10,15}=3)$ |
| * 29 | $\{[14]^1, [4]^7, [-2]^{21}\}$ | 13 28 | $(n_{13}=28,\ n_{28}=1,\ k_{13,28}=1)$ |
| 30 | $\{[16]^1, [4]^7, [-2]^{22}\}$ | 9 12 17 24 | $(n_9=2, n_{12}=9,\ n_{17}=18,\ n_{24}=1)$ |
| * 36 | $\{[21]^1, [5]^7, [-2]^{28}\}$ | 12 18 28 | $(n_{12}=0,\ n_{18}=28,\ n_{28}=8,\ k_{18,28}=6)$ |

vertices of degree 7 induce a coclique, and the vertices of degree 16 induce a (unique) 12-regular graph. Now the fact that any two vertices of degree 16 have 12 common neighbors uniquely determines this graph, and we find the graph found by Bridges and Mena, and which can be described with the points and planes of $AG(3, 2)$.

Next, consider the case on 24 vertices, with $n_7 = 4$, $n_{10} = 16$ and $n_{15} = 4$ (and $n_{22} = 0$). Here we have that $k_{7,7} = 3$, $k_{7,10} = 4$, $k_{7,15} = 0$, $k_{10,7} = 1$, $k_{10,10} = 6$, $k_{10,15} = 3$, $k_{15,7} = 0$, $k_{15,10} = 12$ and $k_{15,15} = 3$. First of all, the vertices of degree 7 induce a clique, and each vertex of degree 10 is adjacent to a unique vertex of degree 7. By using the fact that a vertex of degree 7 and a vertex of degree 10 that are not adjacent have two common neighbors, and a vertex of degree 10 is adjacent to 6 other vertices of degree 10, we find that the 4 neighbors of degree 10 of a vertex of degree 7 induce a clique. Also the vertices of degree 15 induce a clique. Furthermore, a vertex $x$ of degree 7 and a vertex $y$ of degree 15 have three common neighbors (which must be of degree 10). This implies that the unique neighbor $z$ of degree 10 of $x$, which is not adjacent to $y$, has 6 common neighbors with $y$. But these are the remaining three vertices of degree 15 ($k_{10,15} = 3$), and the other three neighbors of degree 10 of $x$. Hence $z$ is adjacent to the remaining three non-neighbors of $y$. Hence the non-neighbors of a vertex of degree 15 also induce a clique. Now it is clear that the induced graph on the vertices of degree 10 is the lattice graph $L_2(4)$, and that the graph we uniquely determined is the one found by Spence, and which is strongly related to the lattice graph $L_2(5)$.

Next, consider the cases with $\theta_1 = 4$. The case on 29 vertices gives a cone over a strongly regular graph, hence over the triangular graph $T(8)$ or one of the three Chang graphs. The case on 30 vertices gives $n_9 = 2$, $n_{12} = 9$, $n_{17} = 18$, $n_{24} = 1$. Now consider the unique vertex $x$ of degree 24. A non-neighbor of $x$ of degree respectively 9, 12, 17, has respectively 4, 8 and 12 common neighbors with $x$. Hence such a non-neighbor is adjacent to respectively 5, 4 and 5 other non-neighbors of $x$. But there are only 5 non-neighbors of $x$, so all non-neighbors of $x$ have degree 12. Now $k_{24,9} = n_9 = 2$, $k_{24,12} = n_{12} - 5 = 4$, and $k_{24,17} = n_{17} = 18$. But now we have a contradiction with the equation $9k_{24,9} + 12k_{24,12} + 17k_{24,17} = 368$ from Section 4.3, hence there can be no such graph.

Finally, consider the case on 36 vertices, which has $n_{18} = 28$ and $n_{28} = 8$ (and $n_{12} = 0$). Now each vertex of degree 28 is adjacent to all 7 other vertices of degree 28, and to 21 vertices of degree 18. Each vertex of degree 18 is adjacent to 6 vertices of degree 28, and to 12 vertices of degree 18. Now it follows that switching with respect to the vertices of degree 28 gives a 14-regular graph with spectrum $\{[14]^1, [5]^8, [-2]^{27}\}$, which must be the triangular graph $T(9)$ (cf. [7, Theorem 6.16]). Hence a graph with spectrum $\{[21]^1, [5]^7, [-2]^{28}\}$ is obtained from switching with respect to an 8-clique in $T(9)$. ∎

## 6. MORE EXAMPLES

After investigating feasible parameter sets for graphs with three eigenvalues, with precisely two vertex degrees (and a regular partition) we constructed the following examples. From the strongly regular (40, 12, 2, 4) graphs we found several graphs on 39 vertices with spectra $\{[14]^1, [2]^{23}, [-4]^{15}\}$ and $\{[20]^1, [2]^{22}, [-4]^{16}\}$. Take such a strongly regular graph, and fix a vertex for which the neighbors induce a graph that is the union of 3-cycles and 6-cycles. This implies that we can partition the neighbors into two sets of size 6 such that any two vertices in different sets are not adjacent. Now delete the fixed vertex, and add edges in between the two sets of size 6 (so adding the edges of a $K_{6,6}$). It is not hard to prove that these graphs have spectrum $\{[14]^1, [2]^{23}, [-4]^{15}\}$. According to Spence [private communication], who classified all (28) strongly regular (40, 12, 2, 4) graphs, 117 graphs are found in the way described. By computer Spence also found 3 graphs with the same spectrum which are not constructed this way, but which also have vertex degrees 12 and 17. From the graphs thus found, we also obtain 120 graphs with spectrum $\{[20]^1, [2]^{22}, [-4]^{16}\}$ by switching with respect to the set of vertices of degree 12 (that is, by interchanging the edges and nonedges between the set of vertices of degree 12 and the set of vertices of degree 17).

Consider a 2-(45, 12, 3) design with a polarity with 36 absolute points. This is equivalent to the property that the design has a symmetric incidence matrix with 36 ones on the diagonal. Now take this incidence matrix minus its diagonal as adjacency matrix of a graph. If the graph has the further property that the 9 nonabsolute points induce a coclique in this graph, then after replacing this coclique by a clique, we find a graph with spectrum $\{[14]^1, [2]^{27}, [-4]^{17}\}$ with 36 vertices of degree 11 and 9 vertices of degree 20. According to Spence [private communication], 2-(45, 12, 3) designs with the required properties exist, and in fact he found that there are precisely 9 corresponding graphs. Also here switching (with respect to the vertices of degree 11) gives other graphs with three eigenvalues, that is, with spectrum $\{[20]^1, [2]^{26}, [-4]^{18}\}$.

### 6.1. *A Mixture of Antipodal Covers and Symmetric Designs*

Suppose we have an antipodal 4-cover of the complete graph $K_{4t^2}$ with $c_2 = t^2$ (where $t$ is some integer), that is, a graph on $16t^2$ vertices, such that the vertices can be partitioned into $4t^2$ sets of size 4, so-called antipodal classes, with the property that vertices in the same class are at distance three, each vertex is adjacent to precisely one vertex from each class other than its own, two adjacent vertices have $t^2-2$ common neighbors, and two nonadjacent vertices from distinct classes have $t^2$ common neighbors. Such covers are known to exist if $t = 2^i$, $i \geqslant 1$ (cf. [12]). Furthermore, suppose we have a symmetric 2-$(4t^2, 2t^2-t, t^2-t)$ design $D$ (a so-called Menon design), which is also known to exist if $t=2^i$, $i \geqslant 1$ (cf. [1]).

The vertices of the graph $G$ we are going to construct are the vertices and the antipodal classes of the antipodal cover. Two vertices of the cover are adjacent in the new graph if and only if they are adjacent in the cover. All antipodal classes are mutually adjacent. Now in order to define the adjacency between a vertex and an antipodal class of the cover, we represent the points of the design $D$ by the antipodal classes, and we do the same for the blocks. Now an antipodal class is adjacent to a vertex of the cover if the antipodal class is incident in $D$ with the antipodal class containing the vertex. In this graph the vertices of the cover have degree $k_1=2t^2-t+4t^2-1$, and the antipodal classes have degree $k_2=4t^2-1+4(2t^2-t)$. We claim that $G$ has eigenvalues $\theta_0=8t^2-2t-1$, $\theta_1=2t-1$ and $\theta_2=-2t-1$. Setting $\alpha_1=\sqrt{2t^2-t}$ and $\alpha_2=2\sqrt{2t^2-t}$ gives the Perron–Frobenius eigenvector, from which we find $\theta_0$. By counting common neighbors we find indeed that $G$ has three eigenvalues: any two antipodal classes are adjacent, and in the design $D$ they are incident with $t^2-t$ common blocks, hence they have $4(t^2-t)$ vertices of the cover as common neighbors. Moreover, they have $4t^2-2$ antipodal classes as common neighbors, so that $\lambda_{2,2}=4(t^2-t)+4t^2-2={\alpha_2}^2-2$. Two adjacent vertices of the cover have $t^2-2$ vertices of the cover as common neighbors, and as the two vertices are in two distinct antipodal classes, which thus are incident in $D$ with $t^2-t$ common classes, they also have these $t^2-t$ classes as common neighbors, so that $\lambda_{1,1}=t^2-t+t^2-2={\alpha_1}^2-2$. Two nonadjacent vertices of the cover, which are from distinct classes have $t^2$ vertices of the cover as common neighbors, and $t^2-t$ classes as common neighbors. Two nonadjacent vertices of the cover, which are in the same antipodal class have the same $(2t^2-t)$ antipodal classes as neighbors (and no vertices of the cover as common neighbors), hence we find in both cases that $\mu_{1,1}=2t^2-t=\alpha_1^2$. If a vertex of the cover is adjacent to an antipodal class, then they have $2t^2-t-1$ antipodal classes as common neighbors. Since the antipodal class is incident with $2t^2-t-1$ classes not containing the vertex (and so adjacent to all vertices of these classes), and the vertex is adjacent to precisely one vertex from each such class, they have $2t^2-t-1$ vertices of the cover as common neighbors, showing that $\lambda_{1,2}=2(2t^2-t-1)=\alpha_1\alpha_2-2$. Similarly, if a vertex of the cover is not adjacent to an antipodal class, then they have $2t^2-t$ antipodal classes as common neighbors, and $2t^2-t$ vertices of the cover as common neighbors, hence $\mu_{1,2}=2(2t^2-t)=\alpha_1\alpha_2$. It now follows from the equations in Section 4 that $G$ has three eigenvalues, with $\theta_1+\theta_2=-2$ and $-\theta_1\theta_2=4t^2-1$.

Instead of the design $D$, we could also have taken its complement, a symmetric 2-$(4t^2, 2t^2+t, t^2+t)$ design. This produces a graph with three eigenvalues $\theta_0=8t^2+2t-1$, $\theta_1=2t-1$ and $\theta_2=-2t-1$. This is easily seen by noticing that this graph is obtained by switching in the graph described above, with respect to the subset of vertices consisting of the vertices of the cover.

### 6.2. *Systems of Linked Symmetric Designs*

A system of $l$ linked symmetric 2-$(v, k, \lambda)$ designs is a collection of sets $V_i$, $i=1, ..., l+1$ and an incidence relation between each pair of sets forming a symmetric 2-$(v, k, \lambda)$ design, such that for any $i, j, h$ the number of $x \in V_i$ incident with both $y \in V_j$ and $z \in V_h$ depends only on whether $y$ and $z$ are incident. The incidence graph of such a system has the union of all $V_i$ as vertex set, where two vertices are adjacent if they are in distinct $V_i$ and they are incident in the corresponding design of the system. This graph is one of the relations in a three-class association scheme (cf. [8, 13]).

Now suppose we have a system of $u+2$ linked symmetric 2-$(u^2(u+2), u(u+1), u)$ designs. The incidence graph of this system is regular with degree $(u+2)(u+1)\,u$, any two adjacent vertices have $u+1$ common neighbors, two nonadjacent vertices from distinct $V_i$ have $(u+1)^2$ common neighbors and two (nonadjacent) vertices from the same $V_i$ have $(u+2)\,u$ common neighbors (cf. [13]). The vertices of the graph $G$ we shall construct are the vertices of the incidence graph and the sets $V_i$, $i=1, ..., l+1$. The induced graph on the vertices of the incidence graph is this graph itself. A set $V_i$ is adjacent to all vertices of the incidence graph except the vertices from $V_i$ itself. In this graph the vertices of the incidence graph have degree $k_1=(u+2)(u+1)\,u+u+2$, and the sets $V_i$ have degree $k_2=(u+2)^2\,u^2$. By taking $\alpha_1=\sqrt{(u+2)(u+1)}$ and $\alpha_2=u\sqrt{(u+2)(u+1)}$ we get the Perron–Frobenius eigenvector, from which we find $\theta_0=(u+2)^2\,u$. By counting common neighbors we find indeed that $G$ has three eigenvalues, with $\theta_1=u$, $\theta_2=-(u+2)\,u$. Unfortunately, only for $u=2$, the required system of linked designs is known to exist. In this case there is precisely one such system (cf. [13]).

## 7. PROBLEMS AND REMARKS

Using the same methods as in the proof of Theorem 7 we were able to classify all graphs with less than 30 vertices and three eigenvalues, which are not strongly regular or complete bipartite (and all of them have least eigenvalue $-2$). Here only two (more) cases had to be checked. Table II consists of all cases with at most 30 vertices, besides the ones with least eigenvalue $-2$.

It is interesting to note that all known examples have two or three distinct degrees, and hence the partition of the vertices according to their degrees is regular. A nice problem would be to construct examples with more than three degrees, and if possible, one for which the partition is not regular.

TABLE II

| $v$ | Spectrum | Degrees | | Existence |
|---|---|---|---|---|
| 22 | $\{[7]^1, [1]^{14}, [-3]^7\}$ | 5 21 | $(n_5=21,\ n_{21}=1,\ k_{5,21}=1)$ | No |
| 23 | $\{[11]^1, [1.562]^{11}, [-2.562]^{11}\}$ | 6 12 22 | $(n_6=11, n_{12}=11, n_{22}=1,$ $k_{6,12}=3,\ k_{6,22}=1, k_{12,22}=1)$ | No |
| 30 | $\{[11]^1, [1]^{21}, [-4]^8\}$ | 5 13 | $(n_5=15, n_{13}=15, k_{5,13}=3)$ | No |
| 30 | $\{[16]^1, [1]^{20}, [-4]^9\}$ | 8 20 | $(n_8=15,\ n_{20}=15, k_{8,20}=8)$ | No |
| 30 | $\{[12]^1, [2]^{15}, [-3]^{14}\}$ | 8 14 | $(n_8=15, n_{14}=15, k_{8,14}=4)$ | ? |

There are no examples known of two connected graphs with three eigenvalues, which have the same spectrum, but have distinct sequences of degrees. Note that the connectivity is essential here: the complete bipartite graph $K_{1,4}$ and the disjoint union of a 4-cycle and a single vertex both have spectrum $\{[2]^1, [0]^3, [-2]^1\}$.

Finally we mention the problem of finding an example with nonintegral eigenvalues, which is not strongly regular or complete bipartite.

*Note added in proof.* de Caen, Spence, and the author have found such graphs (on 43 vertices); these will appear in a forthcoming paper.

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## REFERENCES

1. Th. Beth, D. Jungnickel, and H. Lenz, "Design Theory," Wissenschaftsverlag, Mannheim, 1985.
2. W. G. Bridges and R. A. Mena, Multiplicative cones—a family of three eigenvalue graphs, *Aequationes Math.* **22** (1981), 208–214.
3. A. E. Brouwer and J. H. van Lint, Strongly regular graphs and partial geometries, *in* "Enumeration and Design. Proc. Silver Jubilee Conf. on Combinatorics, Waterloo, 1982" (D. M. Jackson and S. A. Vanstone, Eds.), pp. 85–122, Academic Press, San Diego, 1984.
4. F. C. Bussemaker and A. Neumaier, Exceptional graphs with smallest eigenvalue $-2$ and related problems, *Math. Comp.* **59** (1992), 583–608.
5. P. J. Cameron, J. M. Goethals, J. J. Seidel, and E. E. Shult, Line graphs, root systems and elliptic geometry, *J. Algebra* **43** (1976), 305–327.
6. P. J. Cameron and J. H. van Lint, "Designs, Graphs, Codes and Their Links," Cambridge Univ. Press, Cambridge, UK, 1991.
7. D. M. Cvetković, M. Doob, and H. Sachs, "Spectra of Graphs," Deutsch, London, 1979.

8. E. R. van Dam, “Graphs with Few Eigenvalues. An Interplay between Combinatorics and Algebra,” Center Dissertation Series 20, Thesis, Tilburg University, 1996.
9. E. R. van Dam and W. H. Haemers, Graphs with constant $\mu$ and $\bar{\mu}$, *Discrete Math.* **182** (1998), 293–307.
10. R. H. F. Denniston, Enumeration of symmetric designs (25, 9, 3), *in* “Algebraic and Geometric Combinatorics” (E. Mendelsohn, Ed.), Ann. Discrete Math., Vol. 15, pp. 111–127, North-Holland, Amsterdam, 1982.
11. M. Doob, Graphs with a small number of distinct eigenvalues, *Ann. New York Acad. Sci.* **175** (1970), 104–110.
12. C. D. Godsil and A. D. Hensel, Distance-regular covers of the complete graph, *J. Comb. Theory Ser. B* **56** (1992), 205–238.
13. R. Mathon, The systems of linked 2-(16, 6, 2) designs, *Ars Combin.* **11** (1981), 131–148.
14. M. Muzychuk and M. Klin, On graphs with three eigenvalues, *Discrete Math.*, to appear, submitted, 1995.
15. J. J. Seidel, A survey of two-graphs, *in* “Coll. Intern. Teorie Combinatorie,” pp. 481–511, Accademia Nazionale dei Lincei, Roma, 1976.
16. J. J. Seidel, Strongly regular graphs, *in* “Surveys in Combinatorics, Proc. 7th Brit. Comb. Conf.” (B. Bollobás, Ed.), pp. 157–180, Cambridge Univ. Press, Cambridge, UK, 1979.